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Widths and shape-preserving widths of Sobolev-type classes of s-monotone functions

J. Gilewicz[a], V.N. Konovalov[b], D. Leviatan[c,*,1]

[a] *Centre de Physique Théorique, CNRS, Luminy Case 907, 13288 Marseille Cedex 09, France*
[b] *Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv 01601, Ukraine*
[c] *School of Mathematical Sciences, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel*

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Abstract

Let I be a finite interval, $r, n \in \mathbb{N}$, $s \in \mathbb{N}_0$ and $1 \leqslant p \leqslant \infty$. Given a set M, of functions defined on I, denote by $\Delta_+^s M$ the subset of all functions $y \in M$ such that the s-difference $\Delta_\tau^s y(\cdot)$ is nonnegative on I, $\forall \tau > 0$. Further, denote by W_p^r the Sobolev class of functions x on I with the seminorm $\|x^{(r)}\|_{L_p} \leqslant 1$. We obtain the exact orders of the Kolmogorov and the linear widths, and of the shape-preserving widths of the classes $\Delta_+^s W_p^r$ in L_q for $s > r+1$ and $(r, p, q) \neq (1, 1, \infty)$. We show that while the widths of the classes depend in an essential way on the parameter s, which characterizes the shape of functions, the shape-preserving widths of these classes remain asymptotically $\asymp n^{-2}$.

MSC: 41A46

Keywords: Sobolev-type classes; Kolmogorov width; Linear width; s-monotone functions

1. Introduction, preliminaries, and the main result

Let X be a real linear space of vectors x with norm $\|x\|_X$, and W and M be nonempty subsets of X. The deviation of W from M is defined by

$$E(W, M)_X := \sup_{x \in W} \inf_{y \in M} \|x - y\|_X.$$

* Corresponding author. Fax: +972 3 6409357.
E-mail address: leviatan@math.tau.ac.il (D. Leviatan).
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The Kolmogorov n-width of W is defined by

$$d_n(W)_X^{\mathrm{kol}} := \inf_{M^n} E(W, M^n)_X, \quad n \geqslant 0,$$

where the infimum is taken over all affine subsets M^n of dimension $\leqslant n$.

For a nonempty subset $V \subseteq X$, we denote the relative n-width of Kolmogorov-type of the set W, subject to the constraint V, by

$$d_n(W, V)_X^{\mathrm{kol}} := \inf_{M^n} E(W, M^n \cap V)_X, \quad n \geqslant 0,$$

where the infimum is taken over all affine subsets M^n of dimension $\leqslant n$ such that $M^n \cap V \neq \emptyset$. Obviously, $d_n(W, X)_X^{\mathrm{kol}} = d_n(W)_X^{\mathrm{kol}}$ and, in general, $d_n(W, V)_X^{\mathrm{kol}} \geqslant d_n(W)_X^{\mathrm{kol}}$.

If $L \subseteq X$ is a linear subspace, we may look at the deviation of W from continuous linear images of $\mathrm{span}(W)$ into L. Namely, we define

$$E(W, L)_X^{\mathrm{lin}} := \inf_A \sup_{x \in W} \|x - Ax\|_X,$$

where the infimum is taken over all continuous linear operators on $\mathrm{span}(W)$, such that $A : \mathrm{span}\,(W) \mapsto L$.

The linear n-width of W is defined by

$$d_n(W)_X^{\mathrm{lin}} := \inf_{L^n} E(W, L^n)_X^{\mathrm{lin}}, \quad n \geqslant 0,$$

where the infimum is taken over all linear subspaces $L^n \subseteq X$, of dimension $\leqslant n$. Obviously,

$$d_n(W)_X^{\mathrm{kol}} \leqslant d_n(W)_X^{\mathrm{lin}}.$$

Let $I := (-1, 1)$, and let x be a function defined on I. For an integer $s \geqslant 0$, set

$$\Delta_\tau^s x(t) := \begin{cases} \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} x(t + k\tau) & (t, t + s\tau) \subseteq I, \\ 0 & \text{otherwise,} \end{cases}$$

the sth difference of the function x, with step $\tau > 0$. A function x is called s-monotone on I if $\Delta_\tau^s x(t) \geqslant 0$, $t \in I$, for all $\tau > 0$. It is well known that if x is s-monotone on I, $s \geqslant 2$, then $x^{(s-2)}$ is locally absolutely continuous in I, (notation $x^{(s-2)} \in AC_{\mathrm{loc}}(I) =: AC_{\mathrm{loc}}$), and $x^{(s-1)}$ is nondecreasing there (see [1,8,9], for various properties of s-monotone functions). Given a function space X, and $W \subseteq X$, as above, we denote by $\Delta_+^s W$ the subset of s-monotone functions $x \in W$.

Finally, for $1 \leqslant p \leqslant \infty$, we denote by $L_p = L_p(I)$ the usual L_p-spaces, and for an integer $r \geqslant 1$, we denote the Sobolev class

$$W_p^r := W_p^r(I) := \{x \mid x^{(r-1)} \in AC_{\mathrm{loc}},\ \|x^{(r)}\|_{L_p} \leqslant 1\}.$$

For $1 \leqslant q \leqslant \infty$, we call the relative width $d_n(\Delta_+^s W_p^r, \Delta_+^s L_q)_X^{\mathrm{kol}}$, the shape-preserving n-width of the class $\Delta_+^s W_p^r$ in L_q.

The asymptotic behavior of the Kolmogorov and linear n-widths of the Sobolev classes W_p^r in L_q, $1 \leqslant q \leqslant \infty$, is well known. Recently Konovalov and Leviatan, [3], have investigated the behavior of the Kolmogorov and linear widths in L_q of the smaller classes $\Delta_+^s W_p^r$, $0 \leqslant s \leqslant r + 1$. Among the results they proved that in most cases the Kolmogorov and linear n-widths of the smaller classes are asymptotically the same as those of the class W_p^r. Namely, the behavior in the typical case when $0 \leqslant s \leqslant r$, and $1 \leqslant p, q \leqslant \infty$ are such that $(r, p, q) \neq (1, 1, \infty)$, is that if $(r, p) \neq (1, 1)$, and if $(r, p) = (1, 1)$ and $1 \leqslant q \leqslant 2$, then

$$d_n\big(\Delta_+^s W_p^r\big)_{L_q}^{\mathrm{kol}} \asymp d_n(W_p^r)_{L_q}^{\mathrm{kol}} \asymp n^{-r+(\max\{1/p,1/2\}-\max\{1/q,1/2\})_+}, \quad n \geqslant r.$$

Here and in the sequel $(a)_+ := \max\{a, 0\}$ and the notation $a_n \asymp b_n$ means that there exist constants $0 < c_* < c^*$, such that $c_* a_n \leqslant b_n \leqslant c^* a_n$, $\forall n$.

The situation is much different for the class $\Delta_+^{r+1} W_p^r$. Namely (see [3]),

Theorem A. *Let $r \geqslant 1$, and let $1 \leqslant p, q \leqslant \infty$ be such that $(r, p, q) \neq (1, 1, \infty)$. Then*

$$d_n\big(\Delta_+^{r+1} W_p^r\big)_{L_q}^{\mathrm{kol}} \asymp d_n\big(\Delta_+^{r+1} W_p^r\big)_{L_q}^{\mathrm{lin}} \asymp n^{-r-1+\min\{1/q',1/2\}}, \quad n > r,$$

where $1/q + 1/q' = 1$.

Thus, both n-widths of $\Delta_+^{r+1} W_p^r$ are asymptotically much smaller than those of W_p^r.

We are going to show that the new pattern prevails when $s > r + 1$. We will show that the asymptotic order of both n-widths of $\Delta_+^s W_p^r$, $s > r+1$, decrease with s (increasing). Specifically, we prove the following result.

Theorem 1. *Let r and s be integers such that $s > r + 1 \geqslant 2$, and let $1 \leqslant p, q \leqslant \infty$ be such that $(r, p, q) \neq (1, 1, \infty)$. Then*

$$d_n(\Delta_+^s W_p^r)_{L_q}^{\mathrm{kol}} \asymp d_n(\Delta_+^s W_p^r)_{L_q}^{\mathrm{lin}} \asymp n^{-s+\min\{1/q',1/2\}}, \quad n \geqslant s. \tag{1.1}$$

Remarks. (i) It is interesting to note that the upper bounds are achieved by piecewise polynomials of degree $s - 1$, with n prescribed knots, that are elements of $\Delta_+^s C^{s-2}$.

(ii) We emphasize that given $x \in \Delta_+^s W_p^r$, $s > r$, we cannot, in general, guarantee the finiteness of the norms of the derivatives of order $> r$, in any of the spaces L_q. (Still when $p = \infty$, $x^{(r+1)} \in L_1$.) Also note that for $s > r$ the asymptotic orders of the Kolmogorov and linear widths in L_q of the classes $\Delta_+^s W_p^r$ depend in an essential way on the parameter s, which characterizes the shape of the functions, while they do not depend on r and p.

(iii) Perhaps one should also point out that also for $s \leqslant r$, we need $s \leqslant r - 1/p + 1/q$ to guarantee that $x^{(s)} \in L_q$.

Konovalov and Leviatan, [4,5], investigated also the behavior of the shape-preserving widths of the classes $\Delta_+^s W_p^r$ for $0 \leqslant s \leqslant r + 1$. Among the results,

Theorem B. *Let $r \geqslant 1$ and let $1 \leqslant p, q \leqslant \infty$ be such that $(r, p, q) \neq (1, 1, \infty)$. Then if $(r, p) \neq (1, 1)$, and if $(r, p) = (1, 1)$ and $1 \leqslant q \leqslant 2$, we have*

$$d_n\big(\Delta_+^0 W_p^r, \Delta_+^0 L_q\big)_{L_q}^{\mathrm{kol}} \asymp n^{-r+(\max\{1/p,1/2\}-\max\{1/q,1/2\})_+}, \quad n \geqslant r.$$

Also, if $1 \leqslant s \leqslant \min\{2, r\}$, then

$$d_n\big(\Delta_+^1 W_p^r, \Delta_+^1 L_q\big)_{L_q}^{\mathrm{kol}} \asymp n^{-r+(1/p-1/q)_+}, \quad n \geqslant r,$$

and if $s = 2$ *and* $r = 1$, *then*

$$d_n\left(\Delta_+^2 W_p^1, \Delta_+^2 L_q\right)_{L_q}^{\mathrm{kol}} \asymp n^{-1-1/q}, \quad n \geqslant 1.$$

Finally, if $3 \leqslant s \leqslant r$, *then*

$$d_n\left(\Delta_+^s W_p^r, \Delta_+^s L_q\right)_{L_q}^{\mathrm{kol}} \asymp n^{-r+s+1/p-3}, \quad n \geqslant r.$$

On the other hand, it was shown in [5] that,

Theorem C. *Let* $r > 1$, *and let* $1 \leqslant p, q \leqslant \infty$. *Then*

$$d_n\left(\Delta_+^{r+1} W_p^r, \Delta_+^{r+1} L_q\right)_{L_q}^{\mathrm{kol}} \asymp n^{-2}, \quad n > r.$$

We are going to show that for all $s > r + 1$, the shape-preserving widths of $\Delta_+^s W_p^r$ in L_q, asymptotically are independent of any of the parameters, p, q, r and s, and that all are of the asymptotic order n^{-2}. Namely,

Theorem 2. *Let* r *and* s *be integers such that* $s > r + 1$, *and let* $1 \leqslant p, q \leqslant \infty$ *be such that* $(r, p, q) \neq (1, 1, \infty)$. *Then*

$$d_n\left(\Delta_+^s W_p^r, \Delta_+^s L_q\right)_{L_q}^{\mathrm{kol}} \asymp n^{-2}, \quad n \geqslant s. \tag{1.2}$$

Remark. Again, the upper bounds are achieved by piecewise polynomials of degree $s - 1$, with n prescribed knots, that are elements of $\Delta_+^s C^{s-2}$.

The rest of the paper is divided into five sections. First in Section 2 we have some auxiliary lemmas and we reduce the question of the upper bounds to a simpler case. The next three are devoted to proving our claims for the upper bounds, and finally we prove the lower bounds in Section 6.

2. Auxiliary lemmas and reduction to subcollections

For $m \geqslant 2$ and $k \in \mathbb{Z}$ let $(k)_m := \binom{m-2+k}{m-1}$, and note that

$$(k)_m = \psi_m(k), \quad k \geqslant -m + 2, \tag{2.1}$$

where

$$\psi_m(t) := ((m-1)!)^{-1} \prod_{l=1}^{m-1} (t + l - 1), \quad t \in \mathbb{R}, \tag{2.2}$$

and that

$$(k)_m \leqslant k^{m-1}, \quad k \geqslant 1. \tag{2.3}$$

Our first lemma is

Lemma 1. *The following two systems of linear equations are equivalent*

$$\tilde{\omega}_i = \sum_{j=1}^{i} (i-j+1)_m \omega_j, \quad i = 1, \ldots, n, \tag{2.4}$$

and

$$\sum_{k=1}^{i} (-1)^{i-k} \binom{m}{i-k} \tilde{\omega}_k = \omega_i, \quad i = 1, \ldots, n. \tag{2.5}$$

Proof. Let $1 \leqslant i \leqslant n$. We substitute (2.4) in (2.5) and get by (2.1) and (2.2),

$$\begin{aligned}\sum_{k=1}^{i} (-1)^{i-k} \binom{m}{i-k} \tilde{\omega}_k &= \sum_{k=1}^{i} (-1)^{i-k} \binom{m}{i-k} \sum_{j=1}^{k} (k-j+1)_m \omega_j \\ &= \sum_{j=1}^{i} \left(\sum_{k=j}^{i} (-1)^{i-k} \binom{m}{i-k} (k-j+1)_m \right) \omega_j \\ &= \sum_{j=1}^{i} \left(\sum_{k=0}^{i-j} (-1)^{k} \binom{m}{k} (i-k-j+1)_m \right) \omega_j \\ &= \sum_{j=1}^{i} \left(\sum_{k=0}^{i-j} (-1)^{k} \binom{m}{k} \psi_m(i-k-j+1) \right) \omega_j =: \Sigma.\end{aligned}$$

Now, if $i > j$, then

$$\sum_{k=0}^{i-j} (-1)^{k} \binom{m}{k} \psi_m(i-k-j+1) = \sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \psi_m(i-k-j+1) = 0,$$

where the right equality follows since ψ_m is a polynomial of degree $< m$. For the left equality we observe that if $m < k \leqslant i-j$, then $\binom{m}{k} = 0$, and if $1 \leqslant i-j < k \leqslant m$, then $1 > i-j-k+1 \geqslant -m+2$ so that $\psi_m(i-k-j+1) = 0$.

For $i = j$

$$\sum_{k=0}^{i-j} (-1)^{k} \binom{m}{k} \psi_m(i-k-j+1) = 1.$$

Hence, $\Sigma = \omega_i$ and the proof is complete. □

Our next result follows immediately by Lemma 1.

Lemma 2. *Given $a, b \in \mathbb{R}^{n-1}$ such that b has nonzero entries. Let $1 \leqslant p \leqslant \infty$ and $M \geqslant 0$, and let*

$$\Omega_m(b) := \left\{ \omega \in \mathbb{R}^{n-1} \mid \|\tilde{\omega}\|_{\ell_p} \leqslant M, \quad \tilde{\omega}_i := b_i \sum_{j=1}^{i} (i-j+1)_m \omega_j, \quad i = 1, \ldots, n-1 \right\}.$$

Then,

$$\max_{\omega\in\Omega_m(b)} |\langle a,\omega\rangle| = M\|c\|_{\ell_{p'}}, \quad c_i := b_i^{-1}\sum_{k=0}^{n-i-1}(-1)^k\binom{m}{k}a_{i+k}, \quad i=1,\dots,n-1,$$

where $1/p+1/p'=1$, *and where* $\langle a,\omega\rangle := \sum_{i=1}^{n-1} a_i\omega_i$.

Proof. By Lemma 1, the operator $T:\mathbb{R}^{n-1}\mapsto\mathbb{R}^{n-1}$, define by $\tilde\omega = T\omega$, where

$$\tilde\omega_i := b_i\sum_{j=1}^{i}(i-j+1)_m\omega_j, \quad i=1,\dots,n-1,$$

is invertible. Hence,

$$\begin{aligned}\max_{\omega\in\Omega_m(b)} |\langle a,\omega\rangle| &= \max_{\|T\omega\|_{\ell_p}\leqslant M} |\langle a,\omega\rangle| \\ &= \max_{\|\tilde\omega\|_{\ell_p}\leqslant M} |\langle T^{-1^*}a,\tilde\omega\rangle| \\ &= M\|T^{-1^*}a\|_{\ell_{p'}} = M\|c\|_{\ell_{p'}}.\end{aligned}$$

This completes the proof. □

Recall that if x is s-monotone, then $x^{(s-1)}$ is nondecreasing, thus it has left and right derivatives $x_-^{(s-1)}(\tau)$ and $x_+^{(s-1)}(\tau)$, $\tau\in I$. We set $x^{(s-1)}(\tau) := (x_+^{(s-1)}(\tau)+x_-^{(s-1)}(\tau))/2$, and let

$$\pi_s(t;x;\tau) := \sum_{k=0}^{s-1}\frac{x^{(k)}(\tau)}{k!}(t-\tau)^k, \quad t\in I, \tag{2.6}$$

denote the Taylor polynomial of x about τ.

Given $x\in\Delta_+^s W_p^r$, let

$$\tilde x(t) := x(t) - \pi_s(t;x;0), \quad t\in I.$$

Then

$$\tilde x^{(k)}(0) = 0, \quad k=0,\dots,s-1,$$

and it is easy to verify that $\tilde x^{(k)}(t)\geqslant 0$, $t\in I_+ := [0,1)$, and $(-1)^{s-k}\tilde x^{(k)}(t)\geqslant 0$, $t\in I_- := (-1,0]$, $k=0,\dots,s-1$. In particular all derivatives $\tilde x^{(k)}$, $k=0,\dots,s-1$, are nondecreasing on I_+, while on I_- the derivatives $\tilde x^{(k)}$, $k=0,\dots,s-1$, alternate in monotonicity. Moreover, Konovalov and Leviatan have proved in [6, Lemma 2] that

$$\|\tilde x^{(r)}\|_{L_p(I)}\leqslant c\|x^{(r)}\|_{L_p(I)}, \tag{2.7}$$

where $c=c(s,p)$.

Hence, if we restrict our discussion to I_+, and if we are able to construct piecewise polynomials $\sigma_{s,n}(t;\tilde x)$, of degree $s-1$ and with $n\geqslant 1$ knots, such that

$$\|\tilde x(\cdot)-\sigma_{s,n}(\cdot;\tilde x)\|_{L_q(I_+)}\leqslant cn^{-\alpha}\|\tilde x^{(r)}\|_{L_p(I)}, \tag{2.8}$$

where c is an absolute constant independent of n, and $\alpha > 0$, then the same estimates for I_- follow by taking $\sigma_{s,n}(t; \tilde{x}) := (-1)^s \sigma_{s,n}(-t; y)$, $t \in I_-$, where $y(t) := (-1)^s \tilde{x}(-t)$, $t \in I$, since the latter satisfies $y^{(k)}(t) = (-1)^{s-k} \tilde{x}^{(k)}(-t) \geqslant 0$, $t \in I_-$, $k = 0, \dots, s-1$.

Therefore, we fix $n \geqslant 1$, and $\beta \geqslant 1$, and denote

$$t_i := t_{n,i} := t_{\beta,n,i} := \begin{cases} 1 - ((n-i)/n)^\beta, & i = 0, 1, \dots, n, \\ -1 + ((n+i)/n)^\beta, & i = -1, \dots, -n, \end{cases} \tag{2.9}$$

and

$$I_i := I_{n,i} := I_{\beta,n,i} := \begin{cases} [t_{\beta,n,i-1}, t_{\beta,n,i}), & i = 1, \dots, n, \\ (t_{\beta,n,i}, t_{\beta,n,i+1}], & i = -1, \dots, -n. \end{cases}$$

In Sections 3 and 5, we will construct various piecewise polynomials $\sigma_{s,n}(t; \tilde{x})$, $t \in I_+$, of degree s having knots at $t_{n,i}$, $1 \leqslant i < n$, satisfying (2.8) for various α's. Then

$$\sigma_{s,n}(t; \tilde{x}) := \begin{cases} \sigma_{s,n}(t; \tilde{x}), & t \in I_+ \\ (-1)^s \sigma_{s,n}(-t; y), & t \in I_-, \end{cases}$$

satisfies

$$\|\tilde{x}(\cdot) - \sigma_{s,n}(\cdot; \tilde{x})\|_{L_q(I)} \leqslant c n^{-\alpha} \|\tilde{x}^{(r)}\|_{L_p(I)}.$$

Hence, setting

$$\sigma_{s,n}(t; x) := \sigma_{s,n}(t; \tilde{x}) + \pi_s(t; x; 0), \quad t \in I,$$

by virtue of (2.7), yields

$$\begin{aligned} \|x(\cdot) - \sigma_{s,n}(\cdot; x)\|_{L_q(I)} &= \|\tilde{x}(\cdot) - \sigma_{s,n}(\cdot; \tilde{x})\|_{L_q(I)} \\ &\leqslant c \left\| \tilde{x}^{(r)} \right\|_{L_p(I)} n^{-\alpha} \\ &\leqslant c \left\| x^{(r)} \right\|_{L_p(I)} n^{-\alpha}. \end{aligned}$$

If we denote by $\Sigma_{s,n} := \Sigma_{\beta,s,n}(I)$, the space of piecewise polynomials $\sigma : I \to \mathbb{R}$, of order s (of degree $s-1$), with knots at $t_{\beta,n,i}$, $i = \pm 1, \dots, \pm(n-1)$ (for $n = 1$, this is just the space of polynomials of degree $s-1$), then $\dim \Sigma_{s,n} = s(2n-1)$, and for $x \in \Delta_+^s W_p^r$, clearly $\sigma_s(\cdot; x) \in \Sigma_{s,n}$.

Therefore, in Sections 3–5 we are going to assume that $x \in \Delta_+^s W_p^r$ satisfies

$$x^{(k)}(0) = 0, \quad k = 0, \dots, s-1, \tag{2.10}$$

so that, in particular $x^{(k)}(t) \geqslant 0$, $t \in I_+$, and all derivatives $x^{(k)}$, $k = 0, \dots, s-1$, are nondecreasing on I_+.

3. Theorem 1, the upper bounds: crude estimates

In view of the above, this section is devoted to proving that an $x \in \Delta_+^s W_p^r$ which satisfies (2.10) can be well approximated by piecewise polynomials associated with it, in a linear fashion, on I_+. But the estimates will only be best possible for a restricted range of q. Specifically, we will show that there is an s-monotone piecewise polynomial with $2n-2$ prescribed knots $\sigma_{s,n}(\cdot;x)$ (see construction below), such that

$$\|x(\cdot)-\sigma_{s,n}(\cdot;x)\|_{L_q(I_+)} \leqslant cn^{-s+1/q'} =: cn^{-\alpha}, \tag{3.1}$$

where $c=c(s,r,p,q)$. In the next section we will improve this estimate for the range $2<q\leqslant\infty$.

Again, we fix $n\geqslant 1$ and $\beta\geqslant 1$, to be prescribed, and we let t_i be defined by (2.9). Denote

$$\pi_{s,i}(t;x) := \pi_s(t;x;t_{i-1}), \quad t\in I_+, \tag{3.2}$$

where the left-hand side is the Taylor polynomial defined in (2.6), and set

$$\sigma_s(t;x) := \sigma_{s,n}(t;x) := \pi_{s,i}(t;x), \quad t\in I_i,\ \ i=1,\dots,n. \tag{3.3}$$

Let $t\in I_i$, $1\leqslant i\leqslant n-1$. Then integration by parts yields,

$$\begin{aligned} x(t)-\sigma_s(t;x) &= x(t)-\pi_{s,i}(t;x)\\ &= \frac{1}{(s-2)!}\int_{t_{i-1}}^{t}\big(x^{(s-1)}(\tau)-x^{(s-1)}(t_{i-1})\big)(t-\tau)^{s-2}\,d\tau. \end{aligned} \tag{3.4}$$

Hence, by the monotonicity of $x^{(s-1)}$, if

$$\omega_i := x^{(s-1)}(t_i)-x^{(s-1)}(t_{i-1}), \quad i=1,\dots,n-1, \tag{3.5}$$

then it is readily seen that

$$\|x(\cdot)-\sigma_s(\cdot;x)\|_{L_q(I_i)} \leqslant \frac{|I_i|^\alpha}{(s-1)!}\omega_i, \quad i=1,\dots,n-1, \tag{3.6}$$

where α is defined in (3.1).

If $t\in I_n$, then

$$\begin{aligned} x(t)-\sigma_s(t;x) &= x(t)-\pi_{s,n}(t;x)\\ &= x(t)-\sum_{k=0}^{r-1}\frac{x^{(k)}(t_{n-1})}{k!}(t-t_{n-1})^k-\sum_{k=r}^{s-1}\frac{x^{(k)}(t_{n-1})}{k!}(t-t_{n-1})^k\\ &\leqslant x(t)-\sum_{k=0}^{r-1}\frac{x^{(k)}(t_{n-1})}{k!}(t-t_{n-1})^k\\ &= \frac{1}{(r-1)!}\int_{t_{n-1}}^{t}x^{(r)}(\tau)(t-\tau)^{r-1}\,d\tau, \end{aligned}$$

where for the inequality we applied that due to (3.6) and the monotonicity of the derivatives, we have $x^{(k)}(t_{n-1})\geqslant 0$, $k=r,\dots,s-1$, and the last equality is just integration by parts. By Hölder's inequality we obtain

$$\|x(\cdot)-\sigma_s(\cdot;x)\|_{L_q(I_n)} \leqslant \left\|x^{(r)}\right\|_{L_p(I_n)}\frac{|I_n|^{r-1/p+1/q}}{(r-1)!} =: \left\|x^{(r)}\right\|_{L_p(I_n)}\frac{|I_n|^\rho}{(r-1)!}, \tag{3.7}$$

which combined with (3.6) yields for $1 \leqslant q < \infty$,

$$\begin{aligned}
&\|x(\cdot) - \sigma_s(\cdot; x)\|^q_{L_q(I_+)} \\
&\quad \leqslant \sum_{i=1}^{n-1} \left(\frac{|I_i|^\alpha}{(s-1)!} \omega_i \right)^q + \left(\left\| x^{(r)} \right\|_{L_p(I_+)} \frac{|I_n|^\rho}{(r-1)!} \right)^q \\
&\quad \leqslant \left(\sum_{i=1}^{n-1} \frac{|I_i|^\alpha}{(s-1)!} \omega_i + \left\| x^{(r)} \right\|_{L_p(I_+)} \frac{|I_n|^\rho}{(r-1)!} \right)^q .
\end{aligned}$$

Hence

$$\|x(\cdot) - \sigma_s(\cdot; x)\|_{L_q(I_+)} \leqslant c \sum_{i=1}^{n-1} |I_i|^\alpha \omega_i + c \left\| x^{(r)} \right\|_{L_p(I_+)} |I_n|^\rho . \tag{3.8}$$

For $q = \infty$, (3.6) and (3.7) immediately imply (3.8).

We wish to estimate $x^{(r)}(t)$ from below. Let

$$y(\tau) := \sum_{j=2}^{n} \omega_{j-1} (\tau - t_{j-1})^0_+, \quad \tau \in [0, 1],$$

and denote $m := s - r - 2$. Then by virtue of (2.10) and the monotonicity of $x^{(s-1)}$, we have

$$\begin{aligned}
x^{(r)}(t) &= c \int_0^t (t-\tau)^m x^{(s-1)}(\tau)\, d\tau \\
&\geqslant c \int_0^t (t-\tau)^m y(\tau)\, d\tau \\
&= c \int_0^t (t-\tau)^m \sum_{j=2}^{n} \omega_{j-1} (\tau - t_{j-1})^0_+ \, d\tau \\
&= c \sum_{j=2}^{n} \omega_{j-1} \int_{t_{j-1}}^t (t-\tau)^m \, d\tau \\
&= c' \sum_{j=2}^{n} \omega_{j-1} (t - t_{j-1})^{m+1} .
\end{aligned} \tag{3.9}$$

Let $\bar{t}_i := (t_i + t_{i-1})/2$, $i = 2, \ldots, n$, and let $t \in [\bar{t}_i, t_i)$. Then for $2 \leqslant j \leqslant i \leqslant n$, we have

$$\begin{aligned}
t - t_{j-1} &= (t - \bar{t}_i) + (\bar{t}_i - t_{i-1}) + \sum_{k=j}^{i-1} (t_k - t_{k-1}) \\
&\geqslant \frac{1}{2} \sum_{k=j}^{i} (t_k - t_{k-1}) = \frac{1}{2} \sum_{k=j}^{i} |I_k| \\
&\geqslant \frac{1}{2} (i - j + 1) |I_i| ,
\end{aligned}$$

since $|I_1| \geqslant |I_2| \geqslant \cdots |I_n|$.

Substituting in (3.9), we obtain

$$x^{(r)}(t) \geqslant \bar{c} \sum_{j=2}^{i} \big((i-j+1)|I_i|\big)^{m+1} \omega_{j-1}, \quad t \in [\bar{t}_i, t_i),$$

so that for $1 \leqslant p < \infty$,

$$\begin{aligned} \|x^{(r)}\|^p_{L_p(I_+)} &\geqslant \sum_{i=2}^{n} |I_i| \left(\bar{c} \sum_{j=2}^{i} \big((i-j+1)|I_i|\big)^{s-1-r} \omega_{j-1} \right)^p \\ &= \sum_{i=1}^{n-1} \left(\bar{c} |I_{i+1}|^{s-r-1/p'} \sum_{j=1}^{i} (i-j+1)^{s-1-r} \omega_j \right)^p \\ &=: \sum_{i=1}^{n-1} \left(\bar{c} |I_{i+1}|^{\gamma} \sum_{j=1}^{i} (i-j+1)^{s-1-r} \omega_j \right)^p, \end{aligned} \tag{3.10}$$

where $\bar{c} = 2^{-(s-1-r)}((s-1-r)!)^{-1}$.

By virtue of (3.3),

$$(i-j+1)^{s-1-r} \geqslant (i-j+1)_{s-r}, \quad 1 \leqslant j \leqslant i \leqslant n-1,$$

which together with (3.10) implies,

$$\sum_{i=1}^{n-1} \left(\bar{c} |I_{i+1}|^{\gamma} \sum_{j=1}^{i} (i-j+1)_{s-r} \omega_j \right)^p \leqslant \|x^{(r)}\|^p_{L_p(I_+)}. \tag{3.11}$$

Given $\beta \geqslant 1$, straightforward computations show that

$$c_1 (n-i)^{\beta-1} n^{-\beta} \leqslant |I_i| \leqslant c_2 (n-i)^{\beta-1} n^{-\beta}, \quad i = 1, \ldots, n-1, \tag{3.12}$$

where $c_1 = c_1(\beta)$ and $c_2 = c_2(\beta)$, and $|I_n| = n^{-\beta}$. In particular, it follows that

$$|I_{i+1}| \geqslant c_3 |I_i|, \quad i = 1, \ldots, n-1, \tag{3.13}$$

for some $c_3 = c_3(\beta)$.

Then, (3.11) becomes

$$\left(\sum_{i=1}^{n-1} \left(c_* (n-i)^{(\beta-1)\gamma} \sum_{j=1}^{i} (i-j+1)_{s-r} \omega_j \right)^p \right)^{1/p} \leqslant \|x^{(r)}\|_{L_p(I_+)} n^{\beta\gamma}, \tag{3.14}$$

and (3.8) becomes

$$\|x(\cdot) - \sigma_s(\cdot; x)\|_{L_q(I_+)} \leqslant c^* \sum_{i=1}^{n-1} \frac{(n-i)^{(\beta-1)\alpha}}{n^{\beta\alpha}} \omega_i + c^* \|x^{(r)}\|_{L_p(I_+)} n^{-\beta\rho}, \tag{3.15}$$

where $c^* = c^*(\beta, r, s, p, q)$ and $c_* = c_*(\beta, r, s, p, q)$.

We are interested in estimating from above the first sum in (3.15), for all $\omega := (\omega_1, \ldots, \omega_{n-1})$, satisfying the constraint (3.14). Therefore, we are going to apply Lemma 2 with $m := s - r$, $M := \|x^{(r)}\|_{L_p(I_+)} n^{\beta\gamma}$, and the fixed $(n-1)$-tuples $a = (a_1, \ldots, a_{n-1})$ and $b = (b_1, \ldots, b_{n-1})$, where

$$a_i := (n-i)^{(\beta-1)\alpha} n^{-\beta\alpha}, \quad i = 1, \ldots, n-1,$$

and

$$b_i := c_*(n-i)^{(\beta-1)\gamma}, \quad i = 1, \ldots, n-1.$$

To this end, let $\eta > -1$, and $m \geqslant 1$, and let $\mu \in \mathbb{R}$, be such that $\mu \geqslant m$. Then it follows by induction (on m) that,

$$\int_0^1 \cdots \int_0^1 (\mu - \tau_1 - \cdots - \tau_m)^\eta \, d\tau_1 \cdots d\tau_m = \left(\prod_{k=1}^{m} (\eta + k) \right)^{-1} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (\mu - k)^{\eta+m}.$$

Hence, for $1 \leqslant i \leqslant n - s + r$, we get

$$\begin{aligned} &\left| \sum_{k=0}^{s-r} (-1)^k \binom{s-r}{k} (n-i-k)^{(\beta-1)\alpha} \right| \\ &\quad = \left(\prod_{k=1}^{s-r} ((\beta-1)\alpha - k + 1) \right) \\ &\qquad \times \left| \int_0^1 \cdots \int_0^1 (n - i - \tau_1 - \cdots - \tau_{s-r})^{(\beta-1)\alpha - s + r} d\tau_1 \cdots d\tau_{s-r} \right| \\ &\quad \leqslant \left(\prod_{k=1}^{s-r} ((\beta-1)\alpha - k + 1) \right) (n-i)^{(\beta-1)\alpha - s + r}, \end{aligned} \tag{3.16}$$

provided we take

$$\beta \geqslant 1 + (s-r)\alpha^{-1}, \tag{3.17}$$

which is always possible since $\alpha \geqslant \rho > 0$.

For $n - s + r + 1 \leqslant i \leqslant n - 1$, we trivially have

$$\left| \sum_{k=0}^{s-r} (-1)^k \binom{s-r}{k} (n-i-k)_+^{(\beta-1)\alpha} \right| \leqslant 2^{s-r} (s-r)^{(\beta-1)\alpha}, \tag{3.18}$$

where $(n-i-k)_+ := \max\{n-i-k, 0\}$.

With the a_i's and b_i's above, (3.16) and (3.18) imply

$$\left| \sum_{k=0}^{s-r} (-1)^k \binom{s-r}{k} a_{i+k} \right| b_i^{-1} \leqslant c n^{-\beta\alpha} \times \begin{cases} (n-i)^{(\beta-1)\rho - s + r}, & 1 \leqslant i \leqslant n - s + r \\ 1, & n - s + r + 1 \leqslant i \leqslant n - 1. \end{cases} \tag{3.19}$$

Let

$$\beta \geqslant 1 + (s - r)\rho^{-1}, \tag{3.20}$$

so that in particular (3.17) holds, and take $1 < p < \infty$. Then

$$\sum_{i=1}^{n-s+r} \left((n-i)^{(\beta-1)\rho-s+r} \right)^{p'} \leqslant n^{((\beta-1)\rho-s+r)p'+1}.$$

Hence, by (3.19),

$$\sum_{i=1}^{n-1} \left(\left| \sum_{k=0}^{s-r} (-1)^k \binom{s-r}{k} a_{i+k} \right| b_i^{-1} \right)^{p'} \leqslant c n^{-p'\beta\alpha} \left(n^{((\beta-1)\rho-s+r)p'+1} + s - r - 1 \right). \tag{3.21}$$

Similarly, for $p = 1$ it follows from that

$$\max_{1 \leqslant i \leqslant n-1} \left| \sum_{k=0}^{s-r} (-1)^k \binom{s-r}{k} a_{i+k} \right| b_i^{-1} \leqslant c n^{-\beta\alpha} n^{(\beta-1)\rho-s+r}.$$

Observe that for $1 \leqslant p < \infty$

$$n^{-\beta\alpha} n^{(\beta-1)\rho-s+r+1/p'} = n^{-\beta\gamma} n^{-\alpha}.$$

Thus, if we choose β so big that it satisfies (3.20), then (3.21) and Lemma 2 provide an upper bound for the first sum on the right-hand side of (3.15). Namely,

$$\sum_{i=1}^{n-1} \frac{(n-i)^{(\beta-1)\alpha}}{n^{\beta\alpha}} \omega_i \leqslant c \left\| x^{(r)} \right\|_{L_p(I_+)} n^{-\alpha}. \tag{3.22}$$

Since by virtue of (3.20),

$$n^{-\beta\rho} \leqslant n^{-\alpha},$$

we conclude by (3.22) that for $1 \leqslant p < \infty$,

$$\|x(\cdot) - \sigma_s(\cdot; x)\|_{L_q(I_+)} \leqslant c \left\| x^{(r)} \right\|_{L_p(I_+)} n^{-\alpha} = c \left\| x^{(r)} \right\|_{L_p(I_+)} n^{-s+1/q'},$$

where $c = c(\beta, r, s, p, q)$.

The case $p = \infty$ is proved in a similar way, with the obvious modification in (3.10) through (3.14). This completes the proof of (3.1).

As we already alluded to at the end of Section 2, for an arbitrary $x \in \Delta_+^s W_p^r$, clearly $\sigma_s(\cdot; x) \in \Sigma_{s,n}$. Also, by our construction, the mapping $A : \operatorname{span}(\Delta_+^s W_p^r) \to \Sigma_{s,n}$, defined by (3.2) and (3.3), is linear. Thus, it follows that

$$d_n(\Delta_+^s W_p^r)_{L_q}^{\text{kol}} \leqslant d_n(\Delta_+^s W_p^r)_{L_q}^{\text{lin}} \leqslant c n^{-s+1/q'}, \quad n \geqslant s, \quad 1 \leqslant q \leqslant \infty, \tag{3.23}$$

where $c = c(r, s, p, q)$. This proves the upper bound in (1.1) for $1 \leqslant q \leqslant 2$.

4. Theorem 1, the uppers bounds: refined estimates

For $2 < q \leqslant \infty$, we are able to improve the estimates (3.23). We do that in this section, by applying discretization techniques.

In this section $n \geqslant 1$ may vary, so we are going to keep it as an index in the relevant places. We take $x \in \Delta_+^s W_p^r$ satisfying (2.10). Let

$$w_n(t) := w_{\beta,n}(t) := n^{-1}\big(1 - |t| + n^{-\beta}\big)^{(\beta-1)/\beta}, \quad t \in I.$$

We first show that

$$\|\big(x(\cdot) - \sigma_{s,n}(\cdot; x)\big) w_n^{-1/q'}(\cdot)\|_{L_1(I_+)} \leqslant cn^{-\alpha}, \quad n \geqslant 1,$$

where $c = c(\beta, r, s, p, q)$, and where α is defined in (3.1) and $\sigma_{s,n}(\cdot; x)$ is defined in (3.3) (note that it was denoted $\sigma_s(\cdot; x)$ there).

Indeed, it is easy to verify that

$$c_1|I_{n,i}| \leqslant w_n(t) \leqslant c_2|I_{n,i}|, \quad t \in I_{n,i}, \ \ 1 \leqslant i \leqslant n,$$

for some $c_1 = c_1(\beta)$ and $c_2 = c_2(\beta)$.

Let $\omega_{n,i} := \omega_i$, where ω_i is defined by (3.5). Then by virtue of (3.6) with $q = 1$, it readily follows that

$$\begin{aligned}\|\big(x(\cdot) - \sigma_{s,n}(\cdot; x)\big) w_n^{-1/q'}(\cdot)\|_{L_1(I_{n,i})} &\leqslant c|I_{n,i}|^s |I_{n,i}|^{-1/q'} \omega_{n,i} = c|I_{n,i}|^{s-1/q'} \omega_{n,i} \\ &= c|I_{n,i}|^{\alpha} \omega_{n,i}, \quad i = 1, \ldots, n-1,\end{aligned}$$

and

$$\begin{aligned}\|\big(x(\cdot) - \sigma_{s,n}(\cdot; x)\big) w_n^{-1/q'}(\cdot)\|_{L_1(I_{n,n})} &\leqslant c\big\|x^{(r)}\big\|_{L_p(I_+)} |I_{n,n}|^{r-1/p+1} |I_{n,n}|^{-1/q'} \\ &= c\big\|x^{(r)}\big\|_{L_p(I_+)} |I_{n,n}|^{r-1/p+1/q} \\ &= c\big\|x^{(r)}\big\|_{L_p(I_+)} |I_{n,n}|^{\rho},\end{aligned}$$

where $c = c(\beta, r, s, p, q)$. Hence,

$$\|\big(x(\cdot) - \sigma_{s,n}(\cdot; x)\big) w_n^{-1/q'}(\cdot)\|_{L_1(I_+)} \leqslant c \sum_{i=1}^{n-1} |I_{n,i}|^{\alpha} \omega_{n,i} + c\big\|x^{(r)}\big\|_{L_p(I_+)} |I_{n,n}|^{\rho},$$

so that the right-hand side is the same as that of (3.8).

Following the explanation at the end of Section 2 and the proof in Section 3, we conclude that with β satisfying (3.20), for each $x \in \Delta_+^s W_p^r$, there is an appropriate $\sigma_{s,n}(\cdot; x)$ such that

$$\|\big(x(\cdot) - \sigma_{s,n}(\cdot; x)\big) w_n^{-1/q'}(\cdot)\|_{L_1(I)} \leqslant cn^{-\alpha}, \quad n \geqslant 1, \tag{4.1}$$

where $c = c(\beta, r, s, p, q)$.

Let $\Sigma_{+,s,n}$ be the space of piecewise polynomials $\sigma : I_+ \mapsto \mathbb{R}$, which are polynomials of degree $\leqslant s-1$ on the intervals $I_{n,i}$, $i = 1, \dots, n$, with endpoints $t_{n,i}$, $i = 0, \dots, n$, defined in (2.9). Then $\dim(\Sigma_{+,s,n}) = sn$, and $\Sigma_{+,s,n} \subseteq \Sigma_{+,s,2n}$, $n \geqslant 1$.

Define a one-to-one mapping between the spaces $\Sigma_{+,s,n}$ and $\mathbb{R}^{sn}$ by the linear invertible discretization operators

$$T_{+,n} : \Sigma_{+,s,n} \ni \sigma_n \mapsto \tau = (\tau_1, \dots, \tau_{sn}) \in \mathbb{R}^{sn},$$

where

$$\tau_i := (sn)^{-\beta/q}(sn - i + 1)^{(\beta-1)/q}\sigma_n(t_{sn,i-1}), \quad i = 1, \dots, sn.$$

The inverse mapping is

$$T_{+,n}^{-1} : \mathbb{R}^{sn} \ni \tau = (\tau_1, \dots, \tau_{sn}) \mapsto \sigma_n \in \Sigma_{+,s,n},$$

where σ_n is uniquely defined by the interpolation equations

$$\sigma_n(t_{sn,i-1}) := (sn)^{\beta/q}(sn - i + 1)^{-(\beta-1)/q}\tau_i, \quad i = 1, \dots, sn.$$

We will prove that

$$c_1 \|T_{+,n}\sigma_n\|_{l_q^{sn}} \leqslant \|\sigma_n\|_{L_q(I_+)} \leqslant c_2 \|T_{+,n}\sigma_n\|_{l_q^{sn}}, \tag{4.2}$$

where $c_1 = c_1(\beta, s, q)$ and $c_2 = c_2(\beta, s, q)$.

To this end, let

$$\sigma_n t := p_{n,i} t, \quad t \in I_{n,i}, \ \ i = 1, \dots, n,$$

where p_{ni} are polynomials of degree $\leqslant s-1$. Then,

$$\|\sigma_n\|_{L_q(I_+)}^q = \sum_{i=1}^{n} \|p_{n,i}\|_{L_q(I_{n,i})}^q. \tag{4.3}$$

Clearly, $t_{sn,s(i-1)+j-1} \in I_{n,i}$, $j = 1, \dots, s$, and

$$p_{n,i}(t) = \sum_{j=1}^{s} p_{n,i}(t_{sn,s(i-1)+j-1}) l_j(t; I_{n,i}), \quad t \in I_{n,i}, \ \ i = 1, \dots, n,$$

where

$$l_j(t; I_{n,i}) := \prod_{\substack{1 \leqslant k \leqslant s \\ k \neq j}} \frac{t - t_{sn,s(i-1)+k-1}}{t_{sn,s(i-1)+j-1} - t_{sn,s(i-1)+k-1}}, \quad t \in I_{n,i}, \ \ j = 1, \dots, s,$$

are the Lagrange fundamental polynomials of degree $s-1$ on $I_{n,i}$.

It is readily seen that

$$\max_{1 \leqslant j \leqslant s} \|l_j(\cdot; I_{n,i})\|_{L_\infty(I_{n,i})} \leqslant \left(\frac{|I_{n,i}|}{|I_{sn,si}|}\right)^{s-1} \leqslant c, \quad i = 1, \dots, n,$$

where $c = c(\beta, s)$. Hence

$$\|p_{n,i}\|_{L_\infty(I_{n,i})} \leqslant \left(\sum_{j=1}^{s} |p_{n,i}(t_{sn,s(i-1)+j-1})|\right)\left(\max_{1\leqslant j\leqslant s} \|l_j(\cdot; I_{n,i})\|_{L_\infty(I_{n,i})}\right)$$
$$\leqslant cs^{1/q'}\left(\sum_{j=1}^{s} |p_{n,i}(t_{sn,s(i-1)+j-1})|^q\right)^{1/q}.$$

This in turn implies that

$$\|p_{n,i}\|^q_{L_q(I_{n,i})} \leqslant |I_{n,i}|\|p_{n,i}\|^q_{L_\infty(I_{n,i})}$$
$$\leqslant c|I_{n,i}|\sum_{j=1}^{s} |p_{n,i}(t_{sn,s(i-1)+j-1})|^q$$
$$\leqslant c\sum_{j=1}^{s}(sn)^{-\beta}(sn-si+1)^{\beta-1}|p_{n,i}(t_{sn,s(i-1)+j-1})|^q,$$

where we applied (3.12).

Finally, by (4.3) we conclude that

$$\|\sigma_n\|^q_{L_q(I_+)} = \sum_{i=1}^{n} \|p_{n,i}\|^q_{L_q(I_{n,i})}$$
$$\leqslant c\sum_{i=1}^{n}\sum_{j=1}^{s}(sn)^{-\beta}(sn-si+s-j+1)^{\beta-1}|p_{n,i}(t_{sn,s(i-1)+j-1})|^q$$
$$= c\sum_{i=1}^{sn}(sn)^{-\beta}(sn-i+1)^{\beta-1}|\sigma_n(t_{sn,i-1})|^q$$
$$= c\|T_{+,n}\sigma_n\|_{l_q^{sn}},$$

where $c = c(\beta, s, q)$, and the right-hand side of (4.2) is proved.

For the left-hand inequality in (4.2), we first observe that for all polynomials p of degree $\leqslant s-1$ and any interval J we have

$$\|p\|_{L_q(J)} \geqslant c|J|^{1/q}\|p\|_{L_\infty(J)},$$

where $c = c(s, q)$. In particular

$$\|p_{n,i}\|_{L_q(I_{n,i})} \geqslant c|I_{n,i}|^{1/q}\|p_{n,i}\|_{L_\infty(I_{n,i})}, \quad i = 1, \ldots, n, \tag{4.4}$$

where $c = c(s, q)$.

Now for $1 \leqslant i \leqslant n$,

$$\begin{aligned}
&|I_{n,i}|^{1/q} \| p_{n,i} \|_{L_\infty(I_{n,i})} \\
&\quad \geqslant |I_{n,i}|^{1/q} \max_{1 \leqslant j \leqslant s} |p_{n,i}(t_{sn,s(i-1)+j-1})| \\
&\quad \geqslant s^{-1/q'} |I_{n,i}|^{1/q} \left(\sum_{j=1}^{s} |p_{n,i}(t_{sn,s(i-1)+j-1})|^q \right)^{1/q} \\
&\quad \geqslant c \left(\sum_{j=1}^{s} (sn)^{-\beta} (sn - s(i-1) - j + 1)^{\beta - 1} |p_{n,i}(t_{sn,s(i-1)+j-1})|^q \right)^{1/q},
\end{aligned}$$

where $c = c(\beta, s, q)$, and where for the last inequality we applied (3.12) and (3.13). Hence, by (4.3) and (4.4),

$$\begin{aligned}
\|\sigma_n\|^q_{L_q(I_+)} &\geqslant \sum_{i=1}^{n} \|p_{n,i}\|^q_{L_q(I_{n,i})} \\
&\geqslant c \sum_{i=1}^{n} \sum_{j=1}^{s} (sn)^{-\beta} (sn - s(i-1) - j + 1)^{\beta-1} |p_{n,i}(t_{sn,s(i-1)+j-1})|^q \\
&= c \sum_{i=1}^{sn} (sn)^{-\beta} (sn - i + 1)^{\beta-1} |\sigma_n(t_{sn,i-1})|^q \\
&= c \|T_{+,n}\sigma_n\|_{l_q^{sn}},
\end{aligned}$$

where $c = c(\beta, s, q)$, and the left-hand side of (4.2) is proved.

Taking into account that

$$c_1 w_{2n}(t) \leqslant w_n(t) \leqslant c_2 w_{2n}(t), \quad t \in I, \quad n \geqslant 1,$$

where $c_1 = c_1(\beta)$ and $c_2 = c_2(\beta)$, a similar proof (see the proof of (4.1)), yields

$$c_1 \|T_{+,n}\sigma_n\|_{l_1^{sn}} \leqslant \|\sigma_n w_n^{-1/q'}\|_{L_1(I_+)} \leqslant c_2 \|T_{+,n}\sigma_n\|_{l_1^{sn}}, \tag{4.5}$$

where $c_1 = c_1(\beta, s, q)$ and $c_2 = c_2(\beta, s, q)$.

Given $x \in \Delta_+^s W_p^r$ satisfying (2.10), set

$$n_\nu := 2^\nu, \quad \nu \geqslant 0,$$

and denote

$$\delta_{s,n_\nu}(t; x) := \begin{cases} \sigma_{s,1}(t; x), & \nu = 0, \\ \sigma_{s,n_\nu}(t; x) - \sigma_{s,n_{\nu-1}}(t; x), & \nu \geqslant 1, \end{cases} \quad t \in I_+,$$

so that $\delta_{s,n_\nu}(\cdot; x) \in \Sigma_{+,s,n_\nu}$, $\nu \geqslant 0$. Moreover, for $\nu \geqslant 1$, it follows by (3.1) that

$$\begin{aligned}
\|\delta_{s,n_\nu}(\cdot; x) w_{n_\nu}^{-1/q'}(\cdot)\|_{L_1(I_+)} &\leqslant c \big\| \big(x(\cdot) - \sigma_{s,n_\nu}(\cdot; x)\big) w_{n_\nu}^{-1/q'}(\cdot) \big\|_{L_1(I_+)} \\
&\quad + c \big\| \big(x(\cdot) - \sigma_{s,n_{\nu-1}}(\cdot; x)\big) w_{n_{\nu-1}}^{-1/q'}(\cdot) \big\|_{L_1(I_+)} \\
&\leqslant c n_\nu^{-\alpha},
\end{aligned}$$

where $c = c(\beta, r, s, p, q)$.

Hence, by virtue of (4.5), we obtain

$$\|T_{+,n_\nu}\delta_{s,n_\nu}(\cdot;x)\|_{l_1^{sn_\nu}} \leqslant c_0 n_\nu^{-\alpha}, \quad \nu \geqslant 1,$$

where $c_0 = c_0(\beta, r, s, p, q)$, which in turn implies that for each such function x, and every $\nu \geqslant 1$, the image $T_{+,n_\nu}\delta_{s,n_\nu}(\cdot;x)$ belongs to the octahedron

$$c_0 n_\nu^{-\alpha} b_1^{sn_\nu} := \left\{ \tau \mid \tau \in l_1^{sn_\nu}, \|\tau\|_{l_1^{sn_\nu}} \leqslant c_0 n_\nu^{-\alpha} \right\}.$$

Let $\{m_\nu\}_{\nu\geqslant 0}$, be a sequence of integers such that $m_0 = s$ and $m_\nu \leqslant sn_\nu$, $\nu \geqslant 1$. By the definition of the linear n-widths, for arbitrary $c^0 > 1$, there exist subspaces $M^{m_\nu} \subseteq \mathbb{R}^{sn_\nu}$, $\nu \geqslant 0$, and linear mappings $A_{m_\nu} : \mathbb{R}^{sn_\nu} \mapsto M^{m_\nu}$, $\nu \geqslant 0$, such that $M^{m_0} = \mathbb{R}^s$, A_{m_0} the identity map on $\mathbb{R}^s$,

$$\dim(M^{m_\nu}) \leqslant m_\nu, \quad \nu \geqslant 1,$$

and

$$\sup_{\tau \in b_1^{sn_\nu}} \|\tau - A_{m_\nu}\tau\|_{l_q^{sn_\nu}} \leqslant c^0 d_{m_\nu}\left(b_1^{sn_\nu}\right)^{\text{lin}}_{l_q^{sn_\nu}}, \quad \nu \geqslant 1.$$

Hence,

$$\|\tau - A_{m_\nu}\tau\|_{l_q^{sn_\nu}} \leqslant c^* n_\nu^{-s+1/q'} d_{m_\nu}\left(b_1^{sn_\nu}\right)^{\text{lin}}_{l_q^{sn_\nu}}, \quad \tau \in c_0 n_\nu^{-s+1/q'} b_1^{sn_\nu}, \quad \nu \geqslant 1, \tag{4.6}$$

where $c^* = c_0 c^0$.

Now let

$$\Sigma_{+,s,n_\nu} \supseteq \Sigma_{+,s,n_\nu}^{m_\nu} := T_{+,n_\nu}^{-1} M^{m_\nu}, \quad \nu \geqslant 0,$$

Then $\dim(\Sigma_{+,s,n_\nu}^{m_\nu}) = \dim(M^{m_\nu})$, $\nu \geqslant 0$, and if

$$\Sigma_{+,s,n_\mu}^{m_0,\ldots,m_\mu} := \operatorname{span}\left(\bigcup_{\nu=0}^{\mu} \Sigma_{+,s,n_\nu}^{m_\nu}\right), \quad \mu \geqslant 0,$$

then $\Sigma_{+,s,n_\mu}^{m_0,\ldots,m_\mu} \subseteq \Sigma_{+,s,n_\mu}$, $\mu \geqslant 0$, and

$$\dim\left(\Sigma_{+,s,n_\mu}^{m_0,\ldots,m_\mu}\right) \leqslant \sum_{\nu=0}^{\mu} m_\nu, \quad \mu \geqslant 0.$$

Define the linear mappings

$$A_+^{m_\nu} : \Sigma_{+,s,n_\nu} \mapsto \Sigma_{+,s,n_\nu}^{m_\nu}, \quad \nu \geqslant 0,$$

by

$$A_+^{m_\nu}\sigma_{s,n_\nu} := T_{+,n_\nu}^{-1} A_{m_\nu} T_{+,n_\nu}\sigma_{s,n_\nu}, \quad \nu \geqslant 0.$$

Then each $x \in \Delta_+^s W_p^r$ may be expanded on I_+ into

$$x(t) = (x(t) - \sigma_{s,n_\mu}(t;x)) + \sum_{\nu=0}^{\mu} \delta_{s,n_\nu}(t;x), \quad \mu \geqslant 0, \tag{4.7}$$

so we set the linear mappings

$$A_+^{m_0,\ldots,m_\mu} x(t) := \sum_{\nu=0}^{\mu} A_+^{m_\nu} \delta_{s,n_\nu}(t;x), \quad t \in I_+, \ \mu \geqslant 0,$$

and we conclude that $A_+^{m_0,\ldots,m_\mu} x \in \Sigma_{+,s,n_\mu}^{m_0,\ldots,m_\mu}$.

In view of (4.7) we have

$$\begin{aligned} &\|x(\cdot) - A_+^{m_0,\ldots,m_\mu} x(\cdot)\|_{L_q(I_+)} \\ &\quad \leqslant \|x(\cdot) - \sigma_{s,n_\mu}(\cdot;x)\|_{L_q(I_+)} \\ &\qquad + \sum_{\nu=1}^{\mu} \|\delta_{s,n_\nu}(\cdot;x) - T_{+,n_\nu}^{-1} A_{m_\nu} T_{+,n_\nu} \delta_{s,n_\nu}(\cdot;x)\|_{L_q(I_+)}, \end{aligned} \tag{4.8}$$

where we observe that if $\nu = 0$, then

$$\delta_{s,1}(\cdot;x) - T_{+,1}^{-1} A_{m_0} T_{+,1} \delta_{s,1}(\cdot;x) = 0.$$

By virtue of (3.1),

$$\|x(\cdot) - \sigma_{s,n_\mu}(\cdot;x)\|_{L_q(I_+)} \leqslant c n_\mu^{-s+1/q'}, \quad \mu \geqslant 0, \tag{4.9}$$

where $c = c(\beta, r, s, p, q)$. Further, for $\nu \geqslant 1$,

$$\begin{aligned} &T_{+,n_\nu} \left(\delta_{s,n_\nu}(\cdot;x) - T_{+,n_\nu}^{-1} A_{m_\nu} T_{+,n_\nu} \delta_{s,n_\nu}(\cdot;x) \right) \\ &\quad = T_{+,n_\nu} \delta_{s,n_\nu}(\cdot;x) - A_{m_\nu} T_{+,n_\nu} \delta_{s,n_\nu}(\cdot;x). \end{aligned}$$

Hence, by virtue of (4.2) and (4.6),

$$\begin{aligned} &\|\delta_{s,n_\nu}(\cdot;x) - T_{+,n_\nu}^{-1} A_{m_\nu} T_{+,n_\nu} \delta_{s,n_\nu}(\cdot;x)\|_{L_q(I_+)} \\ &\quad \leqslant c_2 \|T_{+,n_\nu} \delta_{s,n_\nu}(\cdot;x) - A_{m_\nu} T_{+,n_\nu} \delta_{s,n_\nu}(\cdot;x)\|_{l_q^{sn_\nu}} \\ &\quad \leqslant c n_\nu^{-\alpha} d_{m_\nu} \left(b_1^{sn_\nu} \right)_{l_q^{sn_\nu}}^{\text{lin}}, \end{aligned} \tag{4.10}$$

where $c = c(\beta, r, s, p, q)$. So we substitute (4.9) and (4.10) into (4.8) to obtain

$$\|x(\cdot) - A_+^{m_0,\ldots,m_\mu} x(\cdot)\|_{L_q(I_+)} \leqslant c n_\mu^{-\alpha} + c \sum_{\nu=1}^{\mu} n_\nu^{-\alpha} d_{m_\nu} \left(b_1^{sn_\nu} \right)_{l_q^{sn_\nu}}^{\text{lin}}, \tag{4.11}$$

where $c = c(\beta, r, s, p, q)$.

Let $\mu \geqslant 1$, and take

$$m_\nu := \begin{cases} s2^\nu & \text{if } \ 0 \leqslant \nu \leqslant \mu, \\ s2^{2\mu-\nu} & \text{if } \ \mu < \nu \leqslant 2\mu. \end{cases} \tag{4.12}$$

Since

$$d_{m_\nu} \left(b_1^{sn_\nu} \right)_{l_q^{sn_\nu}}^{\text{lin}} = d_{sn_\nu} \left(b_1^{sn_\nu} \right)_{l_q^{sn_\nu}}^{\text{lin}} = 0, \quad \nu = 0, \ldots, \mu,$$

it follows by (4.11) that

$$\|x(\cdot) - A_+^{m_0,\ldots,m_{2\mu}} x(\cdot)\|_{L_q(I_+)} \leqslant c n_{2\mu}^{-\alpha} + c \sum_{v=\mu+1}^{2\mu} n_v^{-\alpha} d_{m_v} \left(b_1^{sn_v}\right)_{l_q^{sn_v}}^{\text{lin}}, \tag{4.13}$$

where $c = c(\beta, r, s, p, q)$.

For $2 < q < \infty$, we estimate the widths in (4.13) by [2, Theorem 2], namely,

$$d_{m_v} \left(b_1^{sn_v}\right)_{l_q^{sn_v}}^{\text{lin}} \leqslant \hat{c} n_v^{1/q} m_v^{-1/2}, \quad v = \mu + 1, \ldots, 2\mu,$$

where $\hat{c} = \hat{c}(s, q)$. Hence, we conclude from (4.12) that

$$\begin{aligned} \sum_{v=\mu+1}^{2\mu} n_v^{-\alpha} d_{m_v} \left(b_1^{sn_v}\right)_{l_q^{sn_v}}^{\text{lin}} &\leqslant \hat{c} \sum_{v=\mu+1}^{2\mu} 2^{-\alpha v} (s 2^v)^{1/q} (s 2^{2\mu - v})^{-1/2} \\ &= \hat{c} s^{1/q - 1/2} 2^{-\mu} \sum_{v=\mu+1}^{2\mu} 2^{-(s-3/2)v} \\ &\leqslant \check{c} 2^{-(s-1/2)\mu}, \end{aligned} \tag{4.14}$$

where $\check{c} = \check{c}(s, q)$.

If $q = \infty$, we apply [7, Chapter 14, (7.2)], to obtain

$$d_{m_v} \left(b_1^{sn_v}\right)_{l_\infty^{sn_v}}^{\text{lin}} \leqslant c (\ln(e n_v / m_v)/m_v)^{1/2}, \quad v = \mu + 1, \ldots, 2\mu,$$

where c is an absolute constant. Hence, we conclude from (4.12) that

$$\begin{aligned} &\sum_{v=\mu+1}^{2\mu} n_v^{-\alpha} d_{m_v} \left(b_1^{sn_v}\right)_{l_\infty^{sn_v}}^{\text{lin}} \\ &\quad\leqslant c \sum_{v=\mu+1}^{2\mu} 2^{-(s-1)v} \left(\ln \frac{e s 2^v}{s 2^{2\mu - v}}\right)^{1/2} (s 2^{2\mu - v})^{-1/2} \\ &\quad= c s^{-1/2} 2^{-\mu} \sum_{v=\mu+1}^{2\mu} (1 + (2v - 2\mu) \ln 2)^{1/2} 2^{-(s-3/2)v} \\ &\quad\leqslant c 2^{1/2} s^{-1/2} 2^{-\mu} \sum_{v=\mu+1}^{\infty} (v - \mu + 1)^{1/2} 2^{-(s-3/2)v} \\ &\quad\leqslant c 2^{1/2} s^{-1/2} 2^{-\mu} \int_{\mu+1}^{\infty} (t - \mu + 1)^{1/2} 2^{-(s-3/2)t} \, dt \\ &\quad= c 2^{s-2} s^{-1/2} 2^{-(s-1/2)\mu} \int_2^{\infty} t^{1/2} 2^{-(s-3/2)t} \, dt \\ &\quad= \bar{c} 2^{-(s-1/2)\mu}, \end{aligned} \tag{4.15}$$

where $\bar{c} = \bar{c}(s)$.

Finally, since $s \geqslant 2$, we have

$$2^{-2\alpha\mu} \leqslant 2^{-(s-1/2)\mu}, \quad n \geqslant 1, \;\; 2 < q \leqslant \infty.$$

Hence, combining (4.13)–(4.15), yields

$$\|x(\cdot) - A_{+}^{m_0,\ldots,m_{2\mu}} x(\cdot)\|_{L_q(I_+)} \leqslant c2^{-(s-1/2)\mu}, \quad 2 < q \leqslant \infty,$$

where $c = c(\beta, r, s, p, q)$.

For $\mu = 0$, it follows by (4.9) that

$$\|x(\cdot) - A_{+}^{m_0} x(\cdot)\|_{L_q(I_+)} \leqslant c, \quad 2 < q \leqslant \infty.$$

Thus, we conclude that

$$E(\Delta_{+}^{s} W_p^r, \Sigma_{+,s,n_{2\mu}}^{m_0,\ldots,m_{2\mu}})_{L_q(I_+)}^{\text{lin}} \leqslant c2^{-(s-1/2)\mu}, \quad \mu \geqslant 0, \;\; 2 < q \leqslant \infty,$$

where $c = c(\beta, r, s, p, q)$, while by (4.12),

$$\dim(\Sigma_{+,s,n_{2\mu}}^{m_0,\ldots,m_{2\mu}}) \leqslant \sum_{v=0}^{2\mu} m_v = s(2^{\mu+1} + 2^{\mu} - 2) < 3s2^{\mu}, \quad \mu \geqslant 0.$$

Similarly, we define the subspaces $\Sigma_{-,s,n_{2\mu}}^{m_0,\ldots,m_{2\mu}}$ on the interval I_-, so that

$$\dim(\Sigma_{-,s,n_{2\mu}}^{m_0,\ldots,m_{2\mu}}) < 3s2^{\mu}, \quad \mu \geqslant 0,$$

and

$$E(\Delta_{+}^{s} W_p^r, \Sigma_{-,s,n_{2\mu}}^{m_0,\ldots,m_{2\mu}})_{L_q(I_-)}^{\text{lin}} \leqslant c2^{-(s-1/2)\mu}, \quad \mu \geqslant 0, \;\; 2 < q \leqslant \infty,$$

where $c = c(\beta, r, s, p, q)$.

Given any $\sigma_+ \in \Sigma_{+,s,n_{2\mu}}^{m_0,\ldots,m_{2\mu}}(I_+)$ and $\sigma_- \in \Sigma_{s,n_{2\mu}}^{m_0,\ldots,m_{2\mu}}(I_-)$, we glue them together into

$$\sigma(t) := \begin{cases} \sigma_+(t), & t \in (0,1), \\ (\sigma_+(0) + \sigma_-(0))/2, & t = 0, \\ \sigma_-(t), & t \in (-1,0), \end{cases}$$

and we denote the set of all such piecewise polynomials by $\Sigma_{s,n_{2\mu}}^{m_0,\ldots,m_{2\mu}}$. Then,

$$\dim(\Sigma_{s,n_{2\mu}}^{m_0,\ldots,m_{2\mu}}) \leqslant 6s2^{\mu},$$

and

$$E(\Delta_{+}^{s} W_p^r, \Sigma_{s,n_{2\mu}}^{m_0,\ldots,m_{2\mu}})_{L_q(I)}^{\text{lin}} \leqslant c2^{-(s-1/2)\mu}, \quad \mu \geqslant 0, \;\; 2 < q \leqslant \infty,$$

where $c = c(\beta, r, s, p, q)$. Therefore, a standard technique yields that

$$d_n(\Delta_{+}^{s} W_p^r)_{L_q(I)}^{\text{kol}} \leqslant d_n(\Delta_{+}^{s} W_p^r)_{L_q(I)}^{\text{lin}} \leqslant cn^{-s+1/2}, \quad n \geqslant s, \;\; 2 < q \leqslant \infty,$$

where $c = c(\beta, r, s, p, q)$. This concludes the proof of the improved upper bounds for $2 < q \leqslant \infty$, and completes the proof of the upper bounds in Theorem 1.

5. Theorem 2, the upper bounds

Recall that $s > r + 1 \geqslant 2$. We fix $n \geqslant 1$, and we use the same notation as in the beginning of Section 2, but we will choose β differently.

Given $x \in \Delta_+^s W_p^r$ that satisfies (2.10). For $n = 1$, we take

$$\sigma_{s,1}(t; x) := 0, \quad t \in I.$$

Then by Hölder's inequality we get

$$\|x(\cdot) - \sigma_{s,1}(\cdot; x)\|_{L_q(I)} \leqslant c \left\| x^{(r)} \right\|_{L_p(I)}. \tag{5.1}$$

For $n \geqslant 2$, we denote by $\sigma_2\left(\cdot; x^{(s-2)}\right) := \sigma_{2,n}\left(\cdot; x^{(s-2)}\right)$, the piecewise linear function, with knots t_i, $i = 0, 1, \ldots, n-2$, which interpolates $x^{(s-2)}$ at t_i, $i = 0, \pm 1, \ldots, \pm(n-1)$, defined in (2.9). Recalling that $x^{(s-2)}$ is convex in I, we conclude that so is σ_2, and that

$$0 \leqslant x^{(s-2)}(\tau) - \sigma_2\left(\tau; x^{(s-2)}\right) \leqslant x^{(s-2)}(\tau) - x^{(s-2)}(t_{n-1}), \quad t_{n-1} \leqslant \tau < 1. \tag{5.2}$$

Now we set

$$\sigma_s(t; x) := \sigma_{s,n}(t; x) := \frac{1}{(s-3)!} \int_0^t \sigma_2\left(\tau; x^{(s-2)}\right)(t-\tau)^{s-3}\, d\tau, \quad t \in I_+.$$

which, evidently, is s-monotone in I_+. We will estimate the L_q distance between x and $\sigma_s(\cdot; x)$ in I_+. Integration by parts yields,

$$\begin{aligned} x(t) - \sigma_s(t; x) &= \frac{1}{(s-3)!} \int_0^t \left(x^{(s-2)}(\tau) - \sigma_s^{(s-2)}(\tau; x)\right)(t-\tau)^{s-3}\, d\tau \\ &= \frac{1}{(s-3)!} \int_0^t \left(x^{(s-2)}(\tau) - \sigma_2\left(\tau; x^{(s-2)}\right)\right)(t-\tau)^{s-3}\, d\tau, \end{aligned}$$

so that for $1 \leqslant q < \infty$,

$$\begin{aligned} &\|x(\cdot) - \sigma_s(\cdot; x)\|_{L_q(I_+)}^q \\ &\quad \leqslant c(s,q) \int_0^1 \left(\int_0^t \left| x^{(s-2)}(\tau) - \sigma_2\left(\tau; x^{(s-2)}\right) \right| (t-\tau)^{s-3}\, d\tau \right)^q dt \\ &\quad = c(s,q) \sum_{i=1}^n \int_{I_i} \left(\int_0^t \left| x^{(s-2)}(\tau) - \sigma_2\left(\tau; x^{(s-2)}\right) \right| (t-\tau)^{s-3}\, d\tau \right)^q dt, \end{aligned} \tag{5.3}$$

where $c(s,q) := ((s-3)!)^{-q}$.

Let $t \in I_i$, $1 \leqslant i \leqslant n-1$. Then we have

$$\begin{aligned} &\int_0^t \left| x^{(s-2)}(\tau) - \sigma_2\left(\tau; x^{(s-2)}\right) \right| (t-\tau)^{s-3}\, d\tau \\ &\quad \leqslant \int_0^{t_i} \left| x^{(s-2)}(\tau) - \sigma_2\left(\tau; x^{(s-2)}\right) \right| (t_i-\tau)^{s-3}\, d\tau \\ &\quad \leqslant \sum_{j=1}^i \int_{I_j} \left| x^{(s-2)}(\tau) - \sigma_2\left(\tau; x^{(s-2)}\right) \right| (t_i - t_{j-1})^{s-3}\, d\tau \\ &\quad \leqslant \sum_{j=1}^i |I_j| (t_i - t_{j-1})^{s-3} \left\| x^{(s-2)}(\cdot) - \sigma_2\left(\cdot; x^{(s-2)}\right) \right\|_{L_\infty(I_j)} \end{aligned}$$

$$= \sum_{j=1}^{i} |I_j| \left(\sum_{k=j}^{i} |I_k| \right)^{s-3} \left\| x^{(s-2)}(\cdot) - \sigma_2\left(\cdot; x^{(s-2)}\right) \right\|_{L_\infty(I_j)}. \tag{5.4}$$

Since $x^{(s-1)}$ is nondecreasing in I_+, and σ_2 interpolates $x^{(s-2)}$ at both endpoints of each interval I_i, $1 \leqslant i \leqslant n-1$, it follows by Whitney's inequality that

$$\begin{aligned} \left\| x^{(s-2)}(\cdot) - \sigma_2\left(\cdot; x^{(s-2)}\right) \right\|_{L_\infty(I_j)} &\leqslant \omega_2\left(x^{(s-1)}; I_j; |I_j|\right) \\ &\leqslant |I_j| \omega\left(x^{(s-1)}; I_j; |I_j|\right) \\ &= |I_j| \omega_j, \quad j = 1, \ldots, n-1, \end{aligned} \tag{5.5}$$

where $\omega\left(x^{(s-1)}; I_j; t\right)$ and $\omega_2\left(x^{(s-1)}; I_j; t\right)$ are, respectively, the first and second moduli of smoothness of $x^{(s-1)}$, in I_j, and ω_j was defined in (3.5).

For $t \in I_n$, we similarly obtain,

$$\begin{aligned} &\int_0^t \left| x^{(s-2)}(\tau) - \sigma_2\left(\tau; x^{(s-2)}\right) \right| (t-\tau)^{s-3}\, d\tau \\ &= \int_0^{t_{n-1}} \left| x^{(s-2)}(\tau) - \sigma_2\left(\tau; x^{(s-2)}\right) \right| (t-\tau)^{s-3}\, d\tau \\ &\quad + \int_{t_{n-1}}^t \left| x^{(s-2)}(\tau) - \sigma_2\left(\tau; x^{(s-2)}\right) \right| (t-\tau)^{s-3}\, d\tau \\ &\leqslant \sum_{j=1}^{n-1} |I_j| \left(\sum_{k=j}^{n-1} |I_k| \right)^{s-3} \left\| x^{(s-2)}(\cdot) - \sigma_2\left(\cdot; x^{(s-2)}\right) \right\|_{L_\infty(I_j)} \\ &\quad + \int_{t_{n-1}}^t \left| x^{(s-2)}(\tau) - \sigma_2\left(\tau; x^{(s-2)}\right) \right| (t-\tau)^{s-3}\, d\tau, \end{aligned} \tag{5.6}$$

and by virtue of (5.2), integration by parts yields,

$$\begin{aligned} &\frac{1}{(s-3)!} \int_{t_{n-1}}^t \left| x^{(s-2)}(\tau) - \sigma_2\left(\tau; x^{(s-2)}\right) \right| (t-\tau)^{s-3}\, d\tau \\ &\leqslant \frac{1}{(s-3)!} \int_{t_{n-1}}^t \left(x^{(s-2)}(\tau) - x^{(s-2)}(t_{n-1}) \right) (t-\tau)^{s-3}\, d\tau \\ &= x(t) - \sum_{k=0}^{s-2} \frac{x^{(k)}(t_{n-1})}{k!} (t - t_{n-1})^k \\ &\leqslant x(t) - \sum_{k=0}^{r-1} \frac{x^{(k)}(t_{n-1})}{k!} (t - t_{n-1})^k \\ &= \frac{1}{(r-1)!} \int_{t_{n-1}}^t x^{(r)}(\tau)(t-\tau)^{r-1}\, d\tau \\ &\leqslant \frac{1}{(r-1)!} \left\| x^{(r)} \right\|_{L_p(I_+)} |I_n|^{r-1/p}, \end{aligned} \tag{5.7}$$

where for the second inequality we used the fact that $x^{(k)}(t_{n-1}) \geqslant 0$, $r-1 < k \leqslant s-2$, and for the last inequality we applied Hölder's inequality.

We substitute (5.4) through (5.7) in (5.3), and use $(|a|+|b|)^q \leqslant 2^{q-1}(|a|^q+|b|^q)$, to obtain

$$\|x(\cdot)-\sigma_s(\cdot;x)\|^q_{L_q(I_+)} \leqslant c\left(\sum_{i=1}^{n-1}\left(|I_i|^{1/q}\sum_{j=1}^{i}|I_j|^2\left(\sum_{k=j}^{i}|I_k|\right)^{s-3}\omega_j\right)^q + \left\|x^{(r)}\right\|^q_{L_p(I_+)}|I_n|^{q\rho}\right), \tag{5.8}$$

for some $c=c(s,q)$. Since $|I_1|\geqslant\cdots\geqslant|I_n|$, we have

$$\begin{aligned}\sum_{i=1}^{n-1}&\left(|I_i|^{1/q}\sum_{j=1}^{i}|I_j|^2\left(\sum_{k=j}^{i}|I_k|\right)^{s-3}\omega_j\right)^q\\
&\leqslant\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i}|I_j|^{2+1/q}\left(\sum_{k=j}^{i}|I_k|\right)^{s-3}\omega_j\right)^q\\
&\leqslant\sum_{i=1}^{n-1}\left(\sum_{j=1}^{n-1}|I_j|^{2+1/q}\left(\sum_{k=j}^{n}|I_k|\right)^{s-3}\omega_j\right)^q\\
&=(n-1)\left(\sum_{i=1}^{n-1}|I_i|^{2+1/q}\left(\sum_{j=i}^{n}|I_j|\right)^{s-3}\omega_i\right)^q.\end{aligned}$$

Hence, (5.8) becomes,

$$\|x(\cdot)-\sigma_s(\cdot;x)\|_{L_q(I_+)} \leqslant c(n-1)^{1/q}\sum_{i=1}^{n-1}|I_i|^{2+1/q}\left(\sum_{j=i}^{n}|I_j|\right)^{s-3}\omega_i + c\left\|x^{(r)}\right\|_{L_p(I_+)}|I_n|^{\rho}, \tag{5.9}$$

where $c=c(s,q)$. It is easy to verify that (5.9) is also valid for $q=\infty$.

We now proceed as in Section 2. Applying (3.12), and

$$\sum_{j=i}^{n}|I_j| = t_n - t_{i-1} = n^{-\beta}(n-i+1)^{\beta}, \quad i=1,\ldots,n,$$

it follows from (5.9) that

$$\|x(\cdot)-\sigma_s(\cdot;x)\|_{L_q(I_+)} \leqslant c\sum_{i=1}^{n-1}\frac{(n-i)^{\beta\alpha-2-1/q}}{n^{\beta\alpha-1/q}}\,\omega_i + c\left\|x^{(r)}\right\|_{L_p(I_+)}n^{-\beta\rho}, \tag{5.10}$$

and we have to estimate the first sum on the right-hand side, for all $\omega := (\omega_1,\ldots,\omega_{n-1})$, satisfying the constraint (3.14) (and the analogous one for $p=\infty$). Again, we apply Lemma 2 with $m := s-r$, $M := \left\|x^{(r)}\right\|_{L_p(I_+)}n^{\beta\gamma}$, where γ was defined in (3.10), and the fixed $(n-1)$-tuples $a=(a_1,\ldots,a_{n-1})$ and $b=(b_1,\ldots,b_{n-1})$, where

$$a_i := (n-i)^{\beta\alpha-2-1/q}n^{-\beta\alpha-1/q}, \quad i=1,\ldots,n-1,$$

and

$$b_i := c_*(n-i)^{(\beta-1)\gamma}, \quad i = 1, \dots, n-1.$$

Just as in (3.16) we obtain for $1 \leqslant i \leqslant n-s+r$,

$$\left|\sum_{k=0}^{s-r}(-1)^k \binom{s-r}{k} (n-i-k)^{\beta\alpha-2-1/q}\right| \leqslant \left(\prod_{k=1}^{s-r}(\beta\alpha - 2 - 1/q - k + 1)\right)(n-i)^{\beta\alpha-2-1/q-s+r}, \tag{5.11}$$

provided

$$\beta \geqslant (s-r+2+1/q)\alpha^{-1}. \tag{5.12}$$

Also, for $n-s+r+1 \leqslant i \leqslant n-1$, we trivially have

$$\left|\sum_{k=0}^{s-r}(-1)^k \binom{s-r}{k} (n-i-k)_+^{\beta\alpha-2-1/q}\right| \leqslant 2^{s-r}(s-r)^{\beta\alpha-2-1/q},$$

where $(n-i-k)_+ := \max\{n-i-k, 0\}$.

Therefore, combining with (5.11), we conclude that for every $1 \leqslant i \leqslant n-1$,

$$\left|\sum_{k=0}^{s-r}(-1)^k \binom{s-r}{k} a_{i+k}\right| b_i^{-1} \leqslant cn^{-\beta\alpha+1/q}(n-i)^{\beta\rho-2-1/p'-1/q}. \tag{5.13}$$

Let $1 < p \leqslant \infty$. Then for

$$\beta \geqslant (2+1/p'+1/q)\rho^{-1}, \tag{5.14}$$

$$\left(\sum_{i=1}^{n-1}\left((n-i)^{\beta\rho-2-1/p'-1/q}\right)^{p'}\right)^{1/p'} \leqslant cn^{\beta\rho-2-1/q}, \tag{5.15}$$

and similarly for $p = 1$,

$$\max_{1 \leqslant i \leqslant n-1}(n-i)^{\beta\rho-2-1/p'-1/q} \leqslant cn^{\beta\rho-2-1/q}. \tag{5.16}$$

Observing that for $1 \leqslant p \leqslant \infty$,

$$n^{-\beta\alpha+1/q}n^{\beta\rho-2-1/q}n^{\beta\gamma} = n^{-2},$$

we choose β so big that it satisfies both (5.12) and (5.14), then Lemma 2, (5.13) and (5.15), and (5.16), provide an upper bound for the first sum on the right-hand side of (5.10). Namely,

$$\sum_{i=1}^{n-1}\frac{(n-i)^{\beta\alpha-2-1/q}}{n^{\beta\alpha-1/q}}\,\omega_i \leqslant c\left\|x^{(r)}\right\|_{L_p(I_+)}n^{-2}. \tag{5.17}$$

Finally, for β satisfying (5.14),

$$n^{-\beta\rho} \leqslant n^{-2}. \tag{5.18}$$

Thus, substituting (5.17) and (5.18) into the right-hand side of (5.10), we obtain

$$\|x(\cdot) - \sigma_s(\cdot; x)\|_{L_q(I_+)} \leqslant c\left\|x^{(r)}\right\|_{L_p(I_+)} n^{-2},$$

where $c = c(\beta, r, s, p, q)$.

As explained in Section 2, for an arbitrary $x \in \Delta_+^s W_p^r$ we obtain an appropriate $\sigma_{s,n}(\cdot; x)$ which is clearly in $\Delta_+^s \Sigma_{s,n}$. Thus, it follows that

$$d_n\left(\Delta_+^s W_p^r, \Delta_+^s L_q\right)_{L_q}^{\mathrm{kol}} \leqslant cn^{-2}, \quad n \geqslant s,$$

where $c = c(r, s, p, q)$. This completes our proof of the upper bounds in (1.2).

6. The lower bounds

We begin with

Proof of Theorem 1. If for $l \geqslant 1$, we denote

$$\overset{\circ}{W}{}_p^l : \left\{x \mid x \in W_p^l,\ x^{(k)}(0) = 0,\ k = 0, \ldots, l-1\right\},$$

then, evidently,

$$\Delta_+^s W_p^{s-1} = \Delta_+^s \overset{\circ}{W}{}_p^{s-1} + \Pi_{s-1}, \tag{6.1}$$

where Π_{s-1} is the space of polynomials of degree $< s - 1$.

It clearly follows by (6.1) that

$$d_{n+s-1}\left(\Delta_+^s W_p^{s-1}\right)_{L_q}^{\mathrm{kol}} \leqslant d_n\left(\Delta_+^s \overset{\circ}{W}{}_p^{s-1}\right)_{L_q}^{\mathrm{kol}}, \quad n \geqslant s. \tag{6.2}$$

For $x \in \overset{\circ}{W}{}_p^{s-1}$, integration by parts yields,

$$x^{(r)}(t) = \frac{1}{(s-r-2)!} \int_0^t x^{(s-1)}(\tau)(t-\tau)^{s-r-2}\, d\tau, \quad t \in I,$$

so that by Hölder's inequality, $\left\|x^{(r)}\right\|_{L_p} \leqslant c_* \left\|x^{(s-1)}\right\|_{L_p}$, where $c_* = 2^{1/p}((s-r-2)!)^{-1}$.

Hence

$$\overset{\circ}{W}{}_p^{s-1} \subseteq c_* \overset{\circ}{W}{}_p^r,$$

which implies

$$c_*^{-1} \Delta_+^s \overset{\circ}{W}{}_p^{s-1} \subset \Delta_+^s \overset{\circ}{W}{}_p^r \subset \Delta_+^s W_p^r. \tag{6.3}$$

Thus, by virtue of (6.2)

$$d_n\left(\Delta_+^s W_p^r\right)_{L_q}^{\mathrm{kol}} \geqslant c_*^{-1} d_{n+s-1}\left(\Delta_+^s W_p^{s-1}\right)_{L_q}^{\mathrm{kol}}, \quad n \geqslant s.$$

Now, our lower bound in (1.1), follows immediately by Theorem A. This completes the proof of the lower bounds in (1.1), and altogether concludes the proof of Theorem 1. □

Proof of Theorem 2. Again by (6.1) we have

$$d_{n+s-1}\left(\Delta_+^s W_p^{s-1}, \Delta_+^s L_q\right)_{L_q}^{\mathrm{kol}} \leqslant d_n\left(\Delta_+^s \overset{\circ}{W}{}_p^{s-1}, \Delta_+^s L_q\right)_{L_q}^{\mathrm{kol}}, \quad n \geqslant s,$$

which combined with (6.3) yields

$$d_n\left(\Delta_+^s W_p^r, \Delta_+^s L_q\right)_{L_q}^{\mathrm{kol}} \geqslant c_*^{-1} d_{n+s}\left(\Delta_+^s W_p^{s-1}, \Delta_+^s L_q\right)_{L_q}^{\mathrm{kol}}, \quad n \geqslant s.$$

Now, our lower bound in (1.2), follows immediately by Theorem D. This completes the proof. □

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